

# INVERSE PROBLEMS FOR DIRAC OPERATORS WITH A FINITE NUMBER OF TRANSMISSION CONDITIONS

YALÇIN GÜLDÜ AND MERVE ARSLANTAŞ

**ABSTRACT.** In this paper, we consider a discontinuous Dirac operator depending polynomially on the spectral parameter and a finite number of transmission conditions. We get some properties of eigenvalues and eigenfunctions. Then, we investigate some uniqueness theorems by using Weyl function and some spectral data.

## 1. Introduction

Inverse problems of spectral analysis recover operators by their spectral data. Fundamental and vast studies about the classical Sturm-Liouville, Dirac operators, Schrödinger equation and hyperbolic equations are well studied (see [1-7] and references therein).

Studies where eigenvalue dependent appears not only in the differential equation but also in the boundary conditions have increased in recent years (see [8-16] and corresponding bibliography cited therein). Moreover, boundary conditions which depend linearly and nonlinearly on the spectral parameter are considered in [8,16-20] and [21-27] respectively. Furthermore, boundary value problems with transmission conditions are also increasingly studied. These types of studies introduce qualitative changes in the exploration. Direct and inverse problems for Sturm-Liouville and Dirac operators with transmission conditions are investigated in some papers (see [7, 28-31] and references therein). Then, differential equations with the spectral parameter and transmission conditions arise in heat, mechanics, mass transfer problems, in diffraction problems and in various physical transfer problems (see [18, 28, 32-39] and corresponding bibliography).

More recently, some boundary value problems with eigenparameter in boundary and transmission conditions are spread out to the case of two, more than two or a finite number of transmission in [40-44] and references therein.

The present paper deals with the discontinuous Dirac operator depending polynomially on the spectral parameter and a finite number of transmission conditions. The aim of the present paper is to obtain the asymptotic formulae of eigenvalues, eigenfunctions and to prove some uniqueness theorems. Especially, some parameters of the considered problem can be determined by Weyl function and some spectral data.

---

2000 *Mathematics Subject Classification.* Primary 34A55, Secondary 34B24, 34L05.

*Key words and phrases.* Dirac operator, Eigenvalues, Eigenfunctions, Transmission Conditions, Weyl Function.

We consider a discontinuous boundary value problem  $L$  with function  $\rho(x)$ ;

$$(1) \quad ly := By'(x) + \Omega(x)y(x) = \lambda\rho(x)y(x), \quad x \in \cup_{i=0}^n (\xi_i, \xi_{i+1}) = \Lambda, \quad \xi_0 = a, \quad \xi_{n+1} = b$$

where  $\rho(x) = \begin{cases} \rho_0, & a \leq x < \xi_1 \\ \rho_i, & \xi_i < x < \xi_{i+1}, \quad i = \overline{1, n-1} \text{ and } \rho_i, i = \overline{0, n} \text{ are given positive} \\ \rho_n, & \xi_n < x \leq b \end{cases}$

real numbers;  $\Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & r(x) \end{pmatrix}, p(x), q(x), r(x) \in L_2[\Lambda, \mathbb{R}]$ ;

$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}, \lambda \in \mathbb{C}$  is a real spectral parameter; boundary conditions at the endpoints

$$(2) \quad l_1 y := a_2(\lambda)y_2(a) - a_1(\lambda)y_1(a) = 0$$

$$(3) \quad l_2 y := b_2(\lambda)y_2(b) - b_1(\lambda)y_1(b) = 0$$

with transmission conditions at  $n$  points  $x = \xi_i, i = \overline{1, n}$

$$(4) \quad l_3 y := y_1(\xi_i + 0) - \theta_i y_1(\xi_i - 0) = 0$$

$$(5) \quad l_4 y := y_2(\xi_i + 0) - \theta_i^{-1} y_2(\xi_i - 0) - \gamma_i(\lambda) y_1(\xi_i - 0) = 0$$

where  $\theta_i, \xi_i (i = \overline{1, n})$  are real numbers,  $a_i(\lambda), b_i(\lambda), (i = 1, 2)$  and  $\gamma_i(\lambda)$ , are real coefficients polynomials for  $\lambda$ .

## 2. Operator Formulation and Properties of Spectrum

In this section, we present the space

$H := L_2(\Lambda) \oplus \mathbb{C}^{m_a} \oplus \mathbb{C}^{m_b} \oplus \sum_{i=1}^n \mathbb{C}^{r_i}$  where  $m_a = \max \{\deg a_i(\lambda)\}, m_b = \max \{\deg b_i(\lambda)\},$   
 $r_i = \max \{\deg \gamma_i(\lambda)\}.$  We define the norm on space  $H$  by

$$(6) \quad \|Y\|^2 := \int_a^b \rho(x) \left\{ |y_1(x)|^2 + |y_2(x)|^2 \right\} dx + \sum_{i=1}^{m_a} Y_i^1 \overline{Y_i^1} \\ + \sum_{j=1}^{m_b} Y_j^2 \overline{Y_j^2} + \sum_{j=1}^n \sum_{i=1}^n Y_i^{3_j} \overline{Y_i^{3_j}}$$

for

$Y \in H, Y = (f(x), Y^1, Y^2, Y^3), Y^1 = (Y_1^1, Y_2^1, \dots, Y_{m_a}^1), Y^2 = (Y_1^2, Y_2^2, \dots, Y_{m_b}^2),$   
 $Y^3 = (Y_1^{3_i}, Y_2^{3_i}, \dots, Y_{r_i}^{3_i}) (i = \overline{1, n}).$

Consider the operator  $T$  defined by the domain

$$D(T) = \left\{ Y \in H : \begin{array}{l} i) y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} \in AC(\Lambda), \quad lf \in L_2(\Lambda) \\ ii) y_1(\xi_i + 0) - \theta_i y_1(\xi_i - 0) = 0 \\ iii) Y_1^1 = a_{m_a 2} y_2(a) - a_{m_a 1} y_1(a) \\ iv) Y_1^2 = b_{m_b 2} y_2(b) - b_{m_b 1} y_1(b) \\ v) Y_1^{3_i} = \gamma_{r_i i} y_1(\xi_i - 0) \end{array} \right\}$$

such that

$$\begin{aligned}
TY &:= W = (ly, W^1, W^2, W^{3_i}), \quad W^1 = (W_1^1, W_2^1, \dots, W_{m_a}^1), \\
W^2 &= (W_1^2, W_2^2, \dots, W_{m_b}^2), \quad W^{3_i} = (W_1^{3_i}, W_2^{3_i}, \dots, W_{r_i}^{3_i}, W_{r_{n+1}}^3) \quad (i = \overline{1, n}) \\
W_i^1 &= Y_{i+1}^1 - a_{m_a-i,2}y_2(a) + a_{m_a-i,1}y_1(a), \quad i = \overline{1, m_a-1} \\
W_{m_a}^1 &= -a_{02}y_2(a) + a_{01}y_1(a) \\
W_j^2 &= Y_{j+1}^1 - b_{m_b-j,2}y_2(b) + b_{m_b-j,1}y_1(b), \quad j = \overline{1, m_b-1} \\
W_{m_b}^2 &= -b_{02}y_2(b) + b_{01}y_1(b) \\
W_1^3 &= Y_{K+1}^{3_1} - \gamma_{r_1-k,1}y_1(\xi_1 - 0), \quad k = \overline{1, r_1-1} \\
W_2^3 &= Y_{K+1}^{3_2} - \gamma_{r_2-k,2}y_1(\xi_2 - 0), \quad k = \overline{1, r_2-1} \\
&\vdots \\
W_{r_n}^3 &= Y_{K+1}^{3_n} - \gamma_{r_n-k,n}y_1(\xi_n - 0), \quad k = \overline{1, r_n-1} \\
W_{r_{n+1}}^3 &= -\gamma_{0t}y_1(\xi_t - 0) + y_2(\xi_t + 0) - \theta_i^{-1}y_2(\xi_t - 0), \quad (t = \overline{1, n})
\end{aligned}$$

Thus, we can rewrite the considered problem (1)-(5) in the operator form as  $TY = \lambda Y$ .

We define the solutions

$$\varphi(x, \lambda) = \varphi_i(x, \lambda), \quad x \in (\xi_i, \xi_{i+1}); \quad \psi(x, \lambda) = \psi_i(x, \lambda), \quad x \in (\xi_i, \xi_{i+1}) \quad (i = \overline{0, n})$$

$\varphi_1(x, \lambda) = (\varphi_{i1}(x, \lambda), \varphi_{i2}(x, \lambda))^T$  and  $\psi_1(x, \lambda) = (\psi_{i1}(x, \lambda), \psi_{i2}(x, \lambda))^T$ ,  $(i = \overline{0, n})$  of equation (1) by the initial conditions

$$(7) \quad \varphi_{01}(a, \lambda) = a_2(\lambda), \quad \varphi_{02}(a, \lambda) = a_1(\lambda)$$

and similarly;

$$(8) \quad \psi_{n1}(b, \lambda) = b_2(\lambda), \quad \psi_{n2}(b, \lambda) = b_1(\lambda)$$

respectively.

**Lemma 1** The following asymptotic behaviours hold for  $|\lambda| \rightarrow \infty$ :

$$\begin{aligned}
\varphi_{01}(x, \lambda) &= \begin{cases} a_{m_2 2} \lambda^{m_2} \cos \lambda \rho_0(x-a) + o(\lambda^{m_2} \exp |\operatorname{Im} \lambda| (x-a) \rho_0), & \deg a_2(\lambda) > \deg a_1(\lambda) \\ -a_{m_1 1} \lambda^{m_1} \sin \lambda \rho_0(x-a) + o(\lambda^{m_1} \exp |\operatorname{Im} \lambda| (x-a) \rho_0), & \deg a_1(\lambda) > \deg a_2(\lambda) \\ \lambda^m (a_{m_2 2} \cos \lambda \rho_0(x-a) - a_{m_1 1} \sin \lambda \rho_0(x-a)) \\ + o(\lambda^m \exp |\operatorname{Im} \lambda| (x-a) \rho_0), & \deg a_1(\lambda) = \deg a_2(\lambda) \end{cases} \\
\varphi_{02}(x, \lambda) &= \begin{cases} a_{m_2 2} \lambda^{m_2} \sin \lambda \rho_0(x-a) + o(\lambda^{m_2} \exp |\operatorname{Im} \lambda| (x-a) \rho_0), & \deg a_2(\lambda) > \deg a_1(\lambda) \\ a_{m_1 1} \lambda^{m_1} \cos \lambda \rho_0(x-a) + o(\lambda^{m_1} \exp |\operatorname{Im} \lambda| (x-a) \rho_0), & \deg a_1(\lambda) > \deg a_2(\lambda) \\ \lambda^m (a_{m_1 1} \cos \lambda \rho_0(x-a) + a_{m_2 2} \sin \lambda \rho_0(x-a)) \\ + o(\lambda^m \exp |\operatorname{Im} \lambda| (x-a) \rho_0), & \deg a_1(\lambda) = \deg a_2(\lambda) \end{cases}
\end{aligned}$$

[illegible]

$$\begin{aligned}
& , \dots, \\
\varphi_{n1}(x, \lambda) = & \left\{ \begin{aligned}
& (-1)^n a_{m22} \lambda^{A+m2} \times \left( \prod_{i=1}^n \gamma_{r_i i} \right) \times \cos \lambda \rho_0(\xi_1 - a) \sin \lambda \rho_n(x - \xi_n) \times \\
& \times \left( \prod_{i=2}^n \sin \lambda \rho_{i-1}(\xi_i - \xi_{i-1}) \right) \\
& + o(\lambda^{A+m2} \exp |\operatorname{Im} \lambda| ((b - \xi_n) \rho_n + \sum_{i=1}^n (\xi_i - \xi_{i-1}) \rho_{i-1}), \deg a_2(\lambda) > \deg a_1(\lambda) \\
& (-1)^{n+1} a_{m11} \lambda^{A+m1} \times \left( \prod_{i=1}^n \gamma_{r_i i} \right) \sin \lambda \rho_0(\xi_1 - a) \sin \lambda \rho_n(x - \xi_n) \times \\
& \times \left( \prod_{i=2}^n \sin \lambda \rho_{i-1}(\xi_i - \xi_{i-1}) \right) \\
& + o(\lambda^{A+m1} \exp |\operatorname{Im} \lambda| ((b - \xi_n) \rho_n + \sum_{i=1}^n (\xi_i - \xi_{i-1}) \rho_{i-1}), \deg a_1(\lambda) > \deg a_2(\lambda) \\
& \lambda^{A+m} \times \left( \prod_{i=1}^n \gamma_{r_i i} \right) \times \\
& \times \left\{ \begin{aligned}
& (-1)^n a_{m22} \cos \lambda \rho_0(\xi_1 - a) \sin \lambda \rho_n(x - \xi_n) \times \left( \prod_{i=2}^n \sin \lambda \rho_{i-1}(\xi_i - \xi_{i-1}) \right) \\
& + (-1)^{n+1} a_{m11} \sin \lambda \rho_0(\xi_1 - a) \sin \lambda \rho_n(x - \xi_n) \times \left( \prod_{i=2}^n \sin \lambda \rho_{i-1}(\xi_i - \xi_{i-1}) \right)
\end{aligned} \right\} \\
& + o(\lambda^{A+m} \exp |\operatorname{Im} \lambda| ((b - \xi_n) \rho_n + \sum_{i=1}^n (\xi_i - \xi_{i-1}) \rho_{i-1}), \deg a_1(\lambda) = \deg a_2(\lambda)
\end{aligned} \right.
\end{aligned}$$

$$\varphi_{n2}(x, \lambda) = \left\{ \begin{array}{l} (-1)^{n+1} a_{m22} \lambda^{A+m2} \times \left( \prod_{i=1}^n \gamma_{r_i i} \right) \cos \lambda \rho_0 (\xi_1 - a) \cos \lambda \rho_n (x - \xi_n) \times \\ \times \left( \prod_{i=2}^n \sin \lambda \rho_{i-1} (\xi_i - \xi_{i-1}) \right) \\ + o(\lambda^{A+m2} \exp |\operatorname{Im} \lambda| ((b - \xi_n) \rho_n + \sum_{i=1}^n (\xi_i - \xi_{i-1}) \rho_{i-1}), \deg a_2(\lambda) > \deg a_1(\lambda) \\ \\ (-1)^n a_{m11} \lambda^{A+m1} \times \left( \prod_{i=1}^n \gamma_{r_i i} \right) \sin \lambda \rho_0 (\xi_1 - a) \cos \lambda \rho_n (x - \xi_n) \times \\ \times \left( \prod_{i=2}^n \sin \lambda \rho_{i-1} (\xi_i - \xi_{i-1}) \right) \\ + o(\lambda^{A+m1} \exp |\operatorname{Im} \lambda| ((b - \xi_n) \rho_n + \sum_{i=1}^n (\xi_i - \xi_{i-1}) \rho_{i-1}), \deg a_1(\lambda) > \deg a_2(\lambda) \\ \\ \lambda^{A+m} \times \left( \prod_{i=1}^n \gamma_{r_i i} \right) \times \\ \times \left\{ \begin{array}{l} (-1)^{n+1} a_{m22} \cos \lambda \rho_0 (\xi_1 - a) \cos \lambda \rho_n (x - \xi_n) \times \left( \prod_{i=2}^n \sin \lambda \rho_{i-1} (\xi_i - \xi_{i-1}) \right) \\ + (-1)^n a_{m11} \sin \lambda \rho_0 (\xi_1 - a) \cos \lambda \rho_n (x - \xi_n) \times \left( \prod_{i=2}^n \sin \lambda \rho_{i-1} (\xi_i - \xi_{i-1}) \right) \end{array} \right\} \\ + o(\lambda^{A+m} \exp |\operatorname{Im} \lambda| ((b - \xi_n) \rho_n + \sum_{i=1}^n (\xi_i - \xi_{i-1}) \rho_{i-1}), \deg a_1(\lambda) = \deg a_2(\lambda) \end{array} \right.$$

where  $A = r_1 + r_2 + \dots + r_n$ .

**Lemma 2** The following asymptotic behaviours hold for  $|\lambda| \rightarrow \infty$ :

$$\psi_{n1}(x, \lambda) = \left\{ \begin{array}{l} b_{m42} \lambda^{m4} \cos \lambda \rho_n (x - b) + o(\lambda^{m4} \exp |\operatorname{Im} \lambda| (x - b) \rho_n), \deg b_2(\lambda) > \deg b_1(\lambda) \\ -b_{m31} \lambda^{m3} \sin \lambda \rho_n (x - b) + o(\lambda^{m3} \exp |\operatorname{Im} \lambda| (x - b) \rho_n), \deg b_1(\lambda) > \deg b_2(\lambda) \\ \lambda^{m_b} (b_{m42} \cos \lambda \rho_n (x - b) - b_{m31} \sin \lambda \rho_n (x - b)) \\ + o(\lambda^{m_b} \exp |\operatorname{Im} \lambda| (x - b) \rho_n), \deg b_1(\lambda) = \deg b_2(\lambda) \end{array} \right.$$

$$\psi_{n2}(x, \lambda) = \left\{ \begin{array}{l} b_{m42} \lambda^{m4} \sin \lambda \rho_n (x - b) + o(\lambda^{m4} \exp |\operatorname{Im} \lambda| (x - b) \rho_n), \deg b_2(\lambda) > \deg b_1(\lambda) \\ b_{m31} \lambda^{m3} \cos \lambda \rho_n (x - b) + o(\lambda^{m3} \exp |\operatorname{Im} \lambda| (x - b) \rho_n), \deg b_1(\lambda) > \deg b_2(\lambda) \\ \lambda^{m_b} (b_{m42} \sin \lambda \rho_n (x - b) + b_{m31} \cos \lambda \rho_n (x - b)) \\ + o(\lambda^{m_b} \exp |\operatorname{Im} \lambda| (x - b) \rho_n), \deg b_1(\lambda) = \deg b_2(\lambda) \end{array} \right.$$

$$\psi_{n-1,1}(x, \lambda) = \begin{cases} b_{m_4 2} \gamma_{r_n n} \lambda^{r_n+m_4} \cos \lambda \rho_n (\xi_n - b) \sin \lambda \rho_{n-1} (x - \xi_n) \\ + o(\lambda^{r_n+m_4} \exp |\operatorname{Im} \lambda| ((\xi_n - b) \rho_n + (x - \xi_n) \rho_{n-1}), \deg b_2(\lambda) > \deg b_1(\lambda) \\ - b_{m_3 1} \gamma_{r_n n} \lambda^{r_n+m_3} \sin \lambda \rho_n (\xi_n - b) \sin \lambda \rho_{n-1} (x - \xi_n) \\ + o(\lambda^{r_n+m_3} \exp |\operatorname{Im} \lambda| ((\xi_n - b) \rho_n + (x - \xi_n) \rho_{n-1}), \deg b_1(\lambda) > \deg b_2(\lambda) \\ \gamma_{r_n n} \lambda^{r_n+m_b} (-b_{m_3 1} \sin \lambda \rho_n (\xi_n - b) + b_{m_4 2} \cos \lambda \rho_n (\xi_n - b)) \sin \lambda \rho_{n-1} (x - \xi_n) \\ + o(\lambda^{r_n+m_b} \exp |\operatorname{Im} \lambda| ((\xi_n - b) \rho_n + (x - \xi_n) \rho_{n-1}), \deg b_1(\lambda) = \deg b_2(\lambda) \end{cases}$$

$$\psi_{n-1,2}(x, \lambda) = \begin{cases} -b_{m_4 2} \gamma_{r_n n} \lambda^{r_n+m_4} \cos \lambda \rho_n (\xi_n - b) \cos \lambda \rho_{n-1} (x - \xi_n) \\ + o(\lambda^{r_n+m_4} \exp |\operatorname{Im} \lambda| ((\xi_n - b) \rho_n + (x - \xi_n) \rho_{n-1}), \deg b_2(\lambda) > \deg b_1(\lambda) \\ b_{m_3 1} \gamma_{r_n n} \lambda^{r_n+m_3} \sin \lambda \rho_n (\xi_n - b) \cos \lambda \rho_{n-1} (x - \xi_n) \\ + o(\lambda^{r_n+m_3} \exp |\operatorname{Im} \lambda| ((\xi_n - b) \rho_n + (x - \xi_n) \rho_{n-1}), \deg b_1(\lambda) > \deg b_2(\lambda) \\ \gamma_{r_n n} \lambda^{r_n+m_b} (b_{m_3 1} \sin \lambda \rho_n (\xi_n - b) - b_{m_4 2} \cos \lambda \rho_n (\xi_n - b)) \cos \lambda \rho_{n-1} (x - \xi_n) \\ + o(\lambda^{r_n+m_b} \exp |\operatorname{Im} \lambda| ((\xi_n - b) \rho_n + (x - \xi_n) \rho_{n-1}), \deg b_1(\lambda) = \deg b_2(\lambda) \end{cases}$$

$$\psi_{n-2,1}(x, \lambda) = \begin{cases} -b_{m_4 2} \gamma_{r_n n} \gamma_{r_{n-1} n-1} \lambda^{r_n+r_{n-1}+m_4} \times \\ \times \cos \lambda \rho_n (\xi_n - b) \sin \lambda \rho_{n-1} (\xi_n - \xi_{n-1}) \sin \lambda \rho_{n-2} (x - \xi_{n-1}) \\ + o(\lambda^{r_n+r_{n-1}+m_4} \exp |\operatorname{Im} \lambda| ((\xi_n - b) \rho_n + (\xi_n - \xi_{n-1}) \rho_{n-1} + (x - \xi_{n-1}) \rho_{n-2}), \\ , \deg b_2(\lambda) > \deg b_1(\lambda) \\ b_{m_3 1} \gamma_{r_n n} \gamma_{r_{n-1} n-1} \lambda^{r_n+r_{n-1}+m_3} \times \\ \times \sin \lambda \rho_n (\xi_n - b) \sin \lambda \rho_{n-1} (\xi_n - \xi_{n-1}) \sin \lambda \rho_{n-2} (x - \xi_{n-1}) \\ + o(\lambda^{r_n+r_{n-1}+m_3} \exp |\operatorname{Im} \lambda| ((\xi_n - b) \rho_n + (\xi_n - \xi_{n-1}) \rho_{n-1} + (x - \xi_{n-1}) \rho_{n-2}), \\ , \deg b_1(\lambda) > \deg b_2(\lambda) \\ \gamma_{r_n n} \gamma_{r_{n-1} n-1} \lambda^{r_n+r_{n-1}+m_b} \sin \lambda \rho_{n-2} (x - \xi_{n-1}) \sin \lambda \rho_{n-1} (\xi_n - \xi_{n-1}) \times \\ \times (b_{m_3 1} \sin \lambda \rho_n (\xi_n - b) - b_{m_4 2} \cos \lambda \rho_n (\xi_n - b)) \\ + o(\lambda^{r_n+r_{n-1}+m_b} \exp |\operatorname{Im} \lambda| ((\xi_n - b) \rho_n + (\xi_n - \xi_{n-1}) \rho_{n-1} + (x - \xi_{n-1}) \rho_{n-2}), \\ , \deg b_1(\lambda) = \deg b_2(\lambda) \end{cases}$$

$$\psi_{n-2,2}(x, \lambda) = \begin{cases} b_{m_4 2} \gamma_{r_n n} \gamma_{r_{n-1} n-1} \lambda^{r_n+r_{n-1}+m_4} \times \\ \times \cos \lambda \rho_n (\xi_n - b) \sin \lambda \rho_{n-1} (\xi_n - \xi_{n-1}) \cos \lambda \rho_{n-2} (x - \xi_{n-1}) \\ + o(\lambda^{r_n+r_{n-1}+m_4} \exp |\operatorname{Im} \lambda| ((\xi_n - b) \rho_n + (\xi_n - \xi_{n-1}) \rho_{n-1} + (x - \xi_{n-1}) \rho_{n-2})) \\ , \deg b_2 (\lambda) > \deg b_1 (\lambda) \\ - b_{m_3 1} \gamma_{r_n n} \gamma_{r_{n-1} n-1} \lambda^{r_n+r_{n-1}+m_3} \times \\ \times \sin \lambda \rho_n (\xi_n - b) \sin \lambda \rho_{n-1} (\xi_n - \xi_{n-1}) \sin \lambda \rho_{n-2} (x - \xi_{n-1}) \\ + o(\lambda^{r_n+r_{n-1}+m_3} \exp |\operatorname{Im} \lambda| ((\xi_n - b) \rho_n + (\xi_n - \xi_{n-1}) \rho_{n-1} + (x - \xi_{n-1}) \rho_{n-2})) \\ , \deg b_1 (\lambda) > \deg b_2 (\lambda) \\ \gamma_{r_n n} \gamma_{r_{n-1} n-1} \lambda^{r_n+r_{n-1}+m_b} \cos \lambda \rho_{n-2} (x - \xi_{n-1}) \sin \lambda \rho_{n-1} (\xi_n - \xi_{n-1}) \times \\ \times (-b_{m_3 1} \sin \lambda \rho_n (\xi_n - b) + b_{m_4 2} \cos \lambda \rho_n (\xi_n - b)) \\ + o(\lambda^{r_n+r_{n-1}+m_b} \exp |\operatorname{Im} \lambda| ((\xi_n - b) \rho_n + (\xi_n - \xi_{n-1}) \rho_{n-1} + (x - \xi_{n-1}) \rho_2)) \\ , \deg b_1 (\lambda) = \deg b_2 (\lambda) \end{cases}$$

, \dots,

$$\psi_{11}(x, \lambda) = \begin{cases} (-1)^n b_{m_4 2} \lambda^{r_2+\dots+r_n+m_4} \times \left( \prod_{i=2}^n \gamma_{r_i i} \right) \times \\ \times \cos \lambda \rho_n (\xi_n - b) \sin \lambda \rho_1 (x - \xi_2) \times \left( \prod_{i=3}^n \sin \lambda \rho_{i-1} (\xi_i - \xi_{i-1}) \right) \\ + o(\lambda^{r_2+\dots+r_n+m_4} \exp |\operatorname{Im} \lambda| ((\xi_2 - \xi_1) \rho_1 + \sum_{i=3}^n (\xi_i - \xi_{i-1}) \rho_{i-1})), \deg b_2 (\lambda) > \deg b_1 (\lambda) \\ (-1)^{n-1} b_{m_3 1} \lambda^{r_2+\dots+r_n+m_3} \times \left( \prod_{i=2}^n \gamma_{r_i i} \right) \times \\ \times \sin \lambda \rho_n (\xi_n - b) \sin \lambda \rho_1 (x - \xi_2) \times \left( \prod_{i=3}^n \sin \lambda \rho_{i-1} (\xi_i - \xi_{i-1}) \right) \\ + o(\lambda^{r_2+\dots+r_n+m_3} \exp |\operatorname{Im} \lambda| ((\xi_2 - \xi_1) \rho_1 + \sum_{i=3}^n (\xi_i - \xi_{i-1}) \rho_{i-1})), \deg b_1 (\lambda) > \deg b_2 (\lambda) \\ \lambda^{r_2+\dots+r_n+m_b} \times \left( \prod_{i=2}^n \gamma_{r_i i} \right) \times \sin \lambda \rho_1 (x - \xi_2) \times \\ \times \left( \prod_{i=3}^n \sin \lambda \rho_{i-1} (\xi_i - \xi_{i-1}) \right) \left\{ \begin{array}{l} (-1)^{n-1} b_{m_3 1} \sin \lambda \rho_n (\xi_n - b) \\ + (-1)^n b_{m_4 2} \cos \lambda \rho_n (\xi_n - b) \end{array} \right\} \\ + o(\lambda^{r_2+\dots+r_n+m_b} \exp |\operatorname{Im} \lambda| ((\xi_2 - \xi_1) \rho_1 + \sum_{i=3}^n (\xi_i - \xi_{i-1}) \rho_{i-1})), \deg b_1 (\lambda) = \deg b_2 (\lambda) \end{cases}$$



$$\psi_{12}(x, \lambda) = \left\{ \begin{array}{l} (-1)^{n-1} b_{m_4 2} \lambda^{r_2 + \dots + r_n + m_4} \times \left( \prod_{i=2}^n \gamma_{r_i i} \right) \times \\ \times \cos \lambda \rho_n (\xi_n - b) \cos \lambda \rho_1 (x - \xi_2) \times \left( \prod_{i=3}^n \sin \lambda \rho_{i-1} (\xi_i - \xi_{i-1}) \right) \\ + o(\lambda^{r_2 + \dots + r_n + m_4} \exp |\operatorname{Im} \lambda| ((\xi_2 - \xi_1) \rho_1 + \sum_{i=3}^n (\xi_i - \xi_{i-1}) \rho_{i-1}), \deg b_2(\lambda) > \deg b_1(\lambda) \\ \\ (-1)^n b_{m_3 1} \lambda^{r_2 + \dots + r_n + m_3} \times \left( \prod_{i=2}^n \gamma_{r_i i} \right) \times \\ \times \sin \lambda \rho_n (\xi_n - b) \cos \lambda \rho_1 (x - \xi_2) \times \left( \prod_{i=3}^n \sin \lambda \rho_{i-1} (\xi_i - \xi_{i-1}) \right) \\ + o(\lambda^{r_2 + \dots + r_n + m_3} \exp |\operatorname{Im} \lambda| ((\xi_2 - \xi_1) \rho_1 + \sum_{i=3}^n (\xi_i - \xi_{i-1}) \rho_{i-1}), \deg b_1(\lambda) > \deg b_2(\lambda) \\ \\ \lambda^{r_2 + \dots + r_n + m_b} \times \left( \prod_{i=2}^n \gamma_{r_i i} \right) \times \cos \lambda \rho_1 (x - \xi_2) \times \\ \times \left( \prod_{i=3}^n \sin \lambda \rho_{i-1} (\xi_i - \xi_{i-1}) \right) \left\{ \begin{array}{l} (-1)^n b_{m_3 1} \sin \lambda \rho_n (\xi_n - b) \\ + (-1)^{n-1} b_{m_4 2} \cos \lambda \rho_n (\xi_n - b) \end{array} \right\} \\ + o(\lambda^{r_2 + \dots + r_n + m_b} \exp |\operatorname{Im} \lambda| ((\xi_2 - \xi_1) \rho_1 + \sum_{i=3}^n (\xi_i - \xi_{i-1}) \rho_{i-1}), \deg b_1(\lambda) = \deg b_2(\lambda) \end{array} \right.$$

$$\psi_{01}(x, \lambda) = \left\{ \begin{aligned} & (-1)^{n-1} b_{m_4 2} \lambda^{A+m_4} \times \left( \prod_{i=1}^n \gamma_{r_i i} \right) \times \\ & \times \cos \lambda \rho_n (\xi_n - b) \sin \lambda \rho_0 (x - \xi_1) \times \left( \prod_{i=2}^n \sin \lambda \rho_{i-1} (\xi_i - \xi_{i-1}) \right) \\ & + o(\lambda^{A+m_4} \exp |\operatorname{Im} \lambda| ((\xi_1 - a) \rho_0 + \sum_{i=2}^n (\xi_i - \xi_{i-1}) \rho_{i-1}), \deg b_2(\lambda) > \deg b_1(\lambda) \\ & (-1)^n b_{m_3 1} \lambda^{A+m_3} \times \left( \prod_{i=1}^n \gamma_{r_i i} \right) \times \\ & \times \sin \lambda \rho_n (\xi_n - b) \sin \lambda \rho_0 (x - \xi_1) \times \left( \prod_{i=2}^n \sin \lambda \rho_{i-1} (\xi_i - \xi_{i-1}) \right) \\ & + o(\lambda^{A+m_3} \exp |\operatorname{Im} \lambda| ((\xi_1 - a) \rho_0 + \sum_{i=2}^n (\xi_i - \xi_{i-1}) \rho_{i-1}), \deg b_1(\lambda) > \deg b_2(\lambda) \\ & \lambda^{A+m_b} \times \left( \prod_{i=1}^n \gamma_{r_i i} \right) \times \sin \lambda \rho_0 (x - \xi_1) \times \\ & \times \left( \prod_{i=2}^n \sin \lambda \rho_{i-1} (\xi_i - \xi_{i-1}) \right) \left\{ \begin{aligned} & (-1)^n b_{m_3 1} \sin \lambda \rho_n (\xi_n - b) \\ & + (-1)^{n-1} b_{m_4 2} \cos \lambda \rho_n (\xi_n - b) \end{aligned} \right\} \\ & + o(\lambda^{A+m_b} \exp |\operatorname{Im} \lambda| ((\xi_1 - a) \rho_0 + \sum_{i=2}^n (\xi_i - \xi_{i-1}) \rho_{i-1}), \deg b_1(\lambda) = \deg b_2(\lambda) \end{aligned} \right.$$

$$\psi_{02}(x, \lambda) = \left\{ \begin{array}{l} (-1)^n b_{m_4 2} \lambda^{A+m_4} \times \left( \prod_{i=1}^n \gamma_{r_i i} \right) \times \\ \times \cos \lambda \rho_n (\xi_n - b) \cos \lambda \rho_0 (x - \xi_1) \times \left( \prod_{i=2}^n \sin \lambda \rho_{i-1} (\xi_i - \xi_{i-1}) \right) \\ + o(\lambda^{A+m_4} \exp |\operatorname{Im} \lambda| ((\xi_1 - a) \rho_0 + \sum_{i=2}^n (\xi_i - \xi_{i-1}) \rho_{i-1}), \deg b_2(\lambda) > \deg b_1(\lambda) \\ \\ (-1)^{n-1} b_{m_3 1} \lambda^{A+m_3} \times \left( \prod_{i=1}^n \gamma_{r_i i} \right) \times \\ \times \sin \lambda \rho_n (\xi_n - b) \cos \lambda \rho_0 (x - \xi_1) \times \left( \prod_{i=2}^n \sin \lambda \rho_{i-1} (\xi_i - \xi_{i-1}) \right) \\ + o(\lambda^{A+m_3} \exp |\operatorname{Im} \lambda| ((\xi_1 - a) \rho_0 + \sum_{i=2}^n (\xi_i - \xi_{i-1}) \rho_{i-1}), \deg b_1(\lambda) > \deg b_2(\lambda) \\ \\ \lambda^{A+m_b} \times \left( \prod_{i=1}^n \gamma_{r_i i} \right) \times \cos \lambda \rho_0 (x - \xi_1) \times \\ \times \left( \prod_{i=2}^n \sin \lambda \rho_{i-1} (\xi_i - \xi_{i-1}) \right) \left\{ \begin{array}{l} (-1)^{n-1} b_{m_3 1} \sin \lambda \rho_n (\xi_n - b) \\ + (-1)^n b_{m_4 2} \cos \lambda \rho_n (\xi_n - b) \end{array} \right\} \\ + o(\lambda^{A+m_b} \exp |\operatorname{Im} \lambda| ((\xi_1 - a) \rho_0 + \sum_{i=2}^n (\xi_i - \xi_{i-1}) \rho_{i-1}), \deg b_1(\lambda) = \deg b_2(\lambda) . \end{array} \right.$$

The function

$$\begin{aligned} \Delta(\lambda) = W(\psi, \varphi) &= \psi(x, \lambda) \varphi'(x, \lambda) - \psi'(x, \lambda) \varphi(x, \lambda) \\ &= \psi_1(b, \lambda) \varphi_2(b, \lambda) - \psi_2(b, \lambda) \varphi_1(b, \lambda) \\ &= \psi_1(a, \lambda) \varphi_2(a, \lambda) - \psi_2(a, \lambda) \varphi_1(a, \lambda) \\ &= b_2(\lambda) \varphi_2(b, \lambda) - b_1(\lambda) \varphi_1(b, \lambda) \\ &= a_1(\lambda) \psi_1(a, \lambda) - a_2(\lambda) \psi_2(a, \lambda) \end{aligned} \tag{9}$$

is called characteristic function of the problem  $L$ . Besides,  $\Delta(\lambda)$  is an entire function in  $\lambda$  and it is obvious that zeros of  $\Delta(\lambda)$  coincide with the eigenvalues of problem  $L$ .

**Lemma 3:** The following asymptotic behaviours hold  $|\lambda| \rightarrow \infty$ :

for  $\deg a_2(\lambda) > \deg a_1(\lambda)$ ;

$$\Delta(\lambda) = \begin{cases} (-1)^{n+1} a_{m_2 2} b_{m_4 2} \lambda^{A+m_2+m_4} \times \\ \times \left( \prod_{i=1}^n \gamma_{r_i i} \right) \cos \lambda \rho_0 (\xi_1 - a) \cos \lambda \rho_n (x - \xi_n) \times \left( \prod_{i=2}^n \sin \lambda \rho_{i-1} (\xi_i - \xi_{i-1}) \right) \\ + o(\lambda^{A+m_2+m_4} \exp |\operatorname{Im} \lambda| ((b - \xi_n) \rho_n + \sum_{i=1}^n (\xi_i - \xi_{i-1}) \rho_{i-1}), \deg b_2(\lambda) > \deg b_1(\lambda)) \\ \\ (-1)^{n+1} a_{m_2 2} b_{m_3 1} \lambda^{A+m_2+m_3} \times \\ \times \left( \prod_{i=1}^n \gamma_{r_i i} \right) \cos \lambda \rho_0 (\xi_1 - a) \sin \lambda \rho_n (x - \xi_n) \times \left( \prod_{i=2}^n \sin \lambda \rho_{i-1} (\xi_i - \xi_{i-1}) \right) \\ + o(\lambda^{A+m_2+m_3} \exp |\operatorname{Im} \lambda| ((b - \xi_n) \rho_n + \sum_{i=1}^n (\xi_i - \xi_{i-1}) \rho_{i-1}), \deg b_1(\lambda) > \deg b_2(\lambda)) \\ \\ (-1)^{n+1} \lambda^{A+m_2+m_b} a_{m_2 2} \times \left( \prod_{i=1}^n \gamma_{r_i i} \right) \times \\ \times \cos \lambda \rho_0 (\xi_1 - a) \times \left( \prod_{i=2}^n \sin \lambda \rho_{i-1} (\xi_i - \xi_{i-1}) \right) \left\{ \begin{array}{l} b_{m_4 2} \cos \lambda \rho_n (x - \xi_n) \\ + b_{m_3 1} \sin \lambda \rho_n (x - \xi_n) \end{array} \right\} \\ + o(\lambda^{A+m_2+m_b} \exp |\operatorname{Im} \lambda| ((b - \xi_n) \rho_n + \sum_{i=1}^n (\xi_i - \xi_{i-1}) \rho_{i-1}), \deg b_1(\lambda) = \deg b_2(\lambda)) \end{cases}$$

for  $\deg a_1(\lambda) > \deg a_2(\lambda)$ ;

$$\Delta(\lambda) = \begin{cases} (-1)^n a_{m_1 1} b_{m_4 2} \lambda^{A+m_1+m_4} \times \\ \times \left( \prod_{i=1}^n \gamma_{r_i i} \right) \sin \lambda \rho_0 (\xi_1 - a) \cos \lambda \rho_n (x - \xi_n) \times \left( \prod_{i=2}^n \sin \lambda \rho_{i-1} (\xi_i - \xi_{i-1}) \right) \\ + o(\lambda^{A+m_1+m_4} \exp |\operatorname{Im} \lambda| ((b - \xi_n) \rho_n + \sum_{i=1}^n (\xi_i - \xi_{i-1}) \rho_{i-1}), \deg b_2(\lambda) > \deg b_1(\lambda)) \\ \\ (-1)^n a_{m_1 1} b_{m_3 1} \lambda^{A+m_1+m_3} \times \\ \times \left( \prod_{i=1}^n \gamma_{r_i i} \right) \sin \lambda \rho_0 (\xi_1 - a) \sin \lambda \rho_n (x - \xi_n) \times \left( \prod_{i=2}^n \sin \lambda \rho_{i-1} (\xi_i - \xi_{i-1}) \right) \\ + o(\lambda^{A+m_1+m_3} \exp |\operatorname{Im} \lambda| ((b - \xi_n) \rho_n + \sum_{i=1}^n (\xi_i - \xi_{i-1}) \rho_{i-1}), \deg b_1(\lambda) > \deg b_2(\lambda)) \\ \\ (-1)^n \lambda^{A+m_1+m_b} a_{m_1 1} \times \left( \prod_{i=1}^n \gamma_{r_i i} \right) \times \\ \times \sin \lambda \rho_0 (\xi_1 - a) \times \left( \prod_{i=2}^n \sin \lambda \rho_{i-1} (\xi_i - \xi_{i-1}) \right) \left\{ \begin{array}{l} b_{m_4 2} \cos \lambda \rho_n (x - \xi_n) \\ + b_{m_3 1} \sin \lambda \rho_n (x - \xi_n) \end{array} \right\} \\ + o(\lambda^{A+m_1+m_b} \exp |\operatorname{Im} \lambda| ((b - \xi_n) \rho_n + \sum_{i=1}^n (\xi_i - \xi_{i-1}) \rho_{i-1}), \deg b_1(\lambda) = \deg b_2(\lambda)) \end{cases}$$

for  $\deg a_1(\lambda) = \deg a_2(\lambda)$ ;

$$\Delta(\lambda) = \left\{ \begin{array}{l} \lambda^{A+m+m_4} \times \left( \prod_{i=1}^n \gamma_{r_i i} \right) \times \left( \prod_{i=2}^n \sin \lambda \rho_{i-1} (\xi_i - \xi_{i-1}) \right) \times \\ \times \left( \begin{array}{l} (-1)^{n+1} a_{m_2 2} b_{m_4 2} \cos \lambda \rho_0 (\xi_1 - a) \cos \lambda \rho_n (x - \xi_n) \\ + (-1)^n a_{m_1 1} b_{m_4 2} \sin \lambda \rho_0 (\xi_1 - a) \cos \lambda \rho_n (x - \xi_n) \end{array} \right) \\ + o(\lambda^{A+m+m_4} \exp |\operatorname{Im} \lambda| ((b - \xi_n) \rho_n + \sum_{i=1}^n (\xi_i - \xi_{i-1}) \rho_{i-1}), \deg b_2(\lambda) > \deg b_1(\lambda) \\ \\ \lambda^{A+m+m_3} \times \left( \prod_{i=1}^n \gamma_{r_i i} \right) \times \left( \prod_{i=2}^n \sin \lambda \rho_{i-1} (\xi_i - \xi_{i-1}) \right) \times \\ \times \left( \begin{array}{l} (-1)^{n+1} a_{m_2 2} b_{m_3 1} \cos \lambda \rho_0 (\xi_1 - a) \sin \lambda \rho_n (x - \xi_n) \\ + (-1)^{n+1} a_{m_1 1} b_{m_3 1} \sin \lambda \rho_0 (\xi_1 - a) \sin \lambda \rho_n (x - \xi_n) \end{array} \right) \\ + o(\lambda^{A+m+m_3} \exp |\operatorname{Im} \lambda| ((b - \xi_n) \rho_n + \sum_{i=1}^n (\xi_i - \xi_{i-1}) \rho_{i-1}), \deg b_1(\lambda) > \deg b_2(\lambda) \\ \\ \lambda^{A+m+m_b} \times \left( \prod_{i=1}^n \gamma_{r_i i} \right) \times \left( \prod_{i=2}^n \sin \lambda \rho_{i-1} (\xi_i - \xi_{i-1}) \right) \times \\ \times \left\{ \begin{array}{l} (-1)^{n+1} a_{m_2 2} \cos \lambda \rho_0 (\xi_1 - a) \left( \begin{array}{l} b_{m_4 2} \cos \lambda \rho_n (x - \xi_n) \\ + b_{m_3 1} \sin \lambda \rho_n (x - \xi_n) \end{array} \right) \\ + (-1)^n a_{m_1 1} \sin \lambda \rho_0 (\xi_1 - a) \left( \begin{array}{l} b_{m_4 2} \cos \lambda \rho_n (x - \xi_n) \\ + b_{m_3 1} \sin \lambda \rho_n (x - \xi_n) \end{array} \right) \end{array} \right\} \\ + o(\lambda^{A+m+m_b} \exp |\operatorname{Im} \lambda| ((b - \xi_n) \rho_n + \sum_{i=1}^n (\xi_i - \xi_{i-1}) \rho_{i-1}), \deg b_1(\lambda) = \deg b_2(\lambda) \end{array} \right.$$

where  $A = r_1 + r_2 + \dots + r_n$ .

### 3. Inverse Problems

In this section, we study the inverse problems for the reconstruction of boundary value problem (1)-(5) by Weyl function and spectral data.

We consider the boundary value problem  $\tilde{L}$  which has the same form with  $L$  but with different coefficients  $\tilde{\Omega}(x)$ ,  $\tilde{f}_1(\lambda)$ ,  $\tilde{f}_2(\lambda)$ ,  $\tilde{\varepsilon}_i$ ,  $\tilde{\theta}_i$ ,  $\tilde{\gamma}_i$  such that

$$\tilde{\Omega}(x) = \begin{pmatrix} \tilde{p}(x) & \tilde{q}(x) \\ \tilde{q}(x) & \tilde{r}(x) \end{pmatrix}, f_1(\lambda) := \frac{a_1(\lambda)}{a_2(\lambda)}, f_2(\lambda) := \frac{b_1(\lambda)}{b_2(\lambda)}.$$

If a certain symbol  $\sigma$  denotes an object related to  $L$ , then the symbol  $\tilde{\sigma}$  denotes the corresponding object related to  $\tilde{L}$ .

**Theorem 1** If  $M(\lambda) = \tilde{M}(\lambda)$ , then  $L = \tilde{L}$ , i.e.,  $\Omega(x) = \tilde{\Omega}(x)$ , a.e. and  $f_1(\lambda) = \tilde{f}_1(\lambda)$ ,  $f_2(\lambda) = \tilde{f}_2(\lambda)$ ,  $\varepsilon_i = \tilde{\varepsilon}_i$ ,  $\theta_i = \tilde{\theta}_i$ ,  $\gamma_i(\lambda) = \tilde{\gamma}_i(\lambda)$ ,  $i = \overline{1, n}$ .

**Proof** We introduce a matrix  $P(x, \lambda) = [P_{kj}(x, \lambda)]_{k,j=1,2}$  by the formula

$$(10) \quad P(x, \lambda) \begin{pmatrix} \tilde{\varphi}_1 & \tilde{\Phi}_1 \\ \tilde{\varphi}_2 & \tilde{\Phi}_2 \end{pmatrix} = \begin{pmatrix} \varphi_1 & \Phi_1 \\ \varphi_2 & \Phi_2 \end{pmatrix}$$

or

$$(11) \quad \begin{pmatrix} P_{11}(x, \lambda) & P_{12}(x, \lambda) \\ P_{21}(x, \lambda) & P_{22}(x, \lambda) \end{pmatrix} = \begin{pmatrix} \varphi_1 \tilde{\Phi}_2 - \Phi_1 \tilde{\varphi}_2 & -\varphi_1 \tilde{\Phi}_1 + \Phi_1 \tilde{\varphi}_1 \\ \varphi_2 \tilde{\Phi}_2 - \tilde{\varphi}_2 \Phi_2 & -\varphi_2 \tilde{\Phi}_1 + \tilde{\varphi}_1 \Phi_2 \end{pmatrix}$$

where  $\Phi(x, \lambda) = \frac{\psi(x, \lambda)}{\Delta(\lambda)}$  and  $W(\tilde{\Phi}, \tilde{\varphi}) = 1$ . Thus, we find

$$(12) \quad \begin{aligned} P_{11}(x, \lambda) &= \varphi_1(x, \lambda) \frac{\tilde{\psi}_2(x, \lambda)}{\tilde{\Delta}(x, \lambda)} - \frac{\psi_1(x, \lambda)}{\Delta(\lambda)} \tilde{\varphi}_2(x, \lambda) \\ P_{12}(x, \lambda) &= -\varphi_1(x, \lambda) \frac{\tilde{\psi}_1(x, \lambda)}{\tilde{\Delta}(\lambda)} + \frac{\psi_1(x, \lambda)}{\Delta(\lambda)} \tilde{\varphi}_1(x, \lambda) \\ P_{21}(x, \lambda) &= \varphi_2(x, \lambda) \frac{\tilde{\psi}_2(x, \lambda)}{\tilde{\Delta}(\lambda)} - \frac{\psi_2(x, \lambda)}{\Delta(\lambda)} \tilde{\varphi}_2(x, \lambda) \\ P_{22}(x, \lambda) &= -\varphi_2(x, \lambda) \frac{\tilde{\psi}_1(x, \lambda)}{\tilde{\Delta}(\lambda)} + \frac{\psi_2(x, \lambda)}{\Delta(\lambda)} \tilde{\varphi}_1(x, \lambda) \end{aligned}$$

We get

$$(13) \quad \begin{aligned} P_{11}(x, \lambda) &= \varphi_1(x, \lambda) \tilde{\varphi}_2(x, \lambda) - \tilde{\varphi}_2(x, \lambda) \phi_1(x, \lambda) + \left( \tilde{M}(\lambda) - M(\lambda) \right) \varphi_1(x, \lambda) \tilde{\varphi}_2(x, \lambda) \\ P_{12}(x, \lambda) &= -\varphi_1(x, \lambda) \tilde{\varphi}_1(x, \lambda) + \tilde{\varphi}_1(x, \lambda) \phi_1(x, \lambda) - \left( \tilde{M}(\lambda) - M(\lambda) \right) \varphi_1(x, \lambda) \tilde{\varphi}_1(x, \lambda) \\ P_{21}(x, \lambda) &= \varphi_2(x, \lambda) \tilde{\varphi}_2(x, \lambda) - \tilde{\varphi}_2(x, \lambda) \phi_2(x, \lambda) + \left( \tilde{M}(\lambda) - M(\lambda) \right) \varphi_2(x, \lambda) \tilde{\varphi}_2(x, \lambda) \\ P_{22}(x, \lambda) &= -\varphi_2(x, \lambda) \tilde{\varphi}_1(x, \lambda) + \tilde{\varphi}_1(x, \lambda) \phi_2(x, \lambda) - \left( \tilde{M}(\lambda) - M(\lambda) \right) \tilde{\varphi}_1(x, \lambda) \varphi_2(x, \lambda). \end{aligned}$$

Thus, if  $M(\lambda) \equiv \tilde{M}(\lambda)$  then the functions  $P_{ij}(x, \lambda)$  ( $i, j = 1, 2$ ) are entire in  $\lambda$  for each fixed  $x$ . Moreover, since asymptotic behaviours of  $\varphi_i(x, \lambda)$ ,  $\tilde{\varphi}_i(x, \lambda)$ ,  $\psi_i(x, \lambda)$ ,  $\tilde{\psi}_i(x, \lambda)$  and

for  $\deg a_2(\lambda) > \deg a_1(\lambda)$ ;

$$|\Delta(\lambda)| \geq C_\delta \begin{cases} |\lambda|^{A+m_2+m_4} \exp |\operatorname{Im} \lambda| \left( (b - \xi_n) \rho_n + \sum_{i=1}^n (\xi_i - \xi_{i-1}) \rho_{i-1} \right), & \deg b_2(\lambda) > \deg b_1(\lambda) \\ |\lambda|^{A+m_2+m_3} \exp |\operatorname{Im} \lambda| \left( (b - \xi_n) \rho_n + \sum_{i=1}^n (\xi_i - \xi_{i-1}) \rho_{i-1} \right), & \deg b_1(\lambda) > \deg b_2(\lambda) \\ |\lambda|^{A+m_2+m_b} \exp |\operatorname{Im} \lambda| \left( (b - \xi_n) \rho_n + \sum_{i=1}^n (\xi_i - \xi_{i-1}) \rho_{i-1} \right), & \deg b_1(\lambda) = \deg b_2(\lambda) \end{cases}$$

for  $\deg a_1(\lambda) > \deg a_2(\lambda)$ ;

$$|\Delta(\lambda)| \geq C_\delta \begin{cases} |\lambda|^{A+m_1+m_4} \exp |\operatorname{Im} \lambda| \left( (b - \xi_n) \rho_n + \sum_{i=1}^n (\xi_i - \xi_{i-1}) \rho_{i-1} \right), & \deg b_2(\lambda) > \deg b_1(\lambda) \\ |\lambda|^{A+m_1+m_3} \exp |\operatorname{Im} \lambda| \left( (b - \xi_n) \rho_n + \sum_{i=1}^n (\xi_i - \xi_{i-1}) \rho_{i-1} \right), & \deg b_1(\lambda) > \deg b_2(\lambda) \\ |\lambda|^{A+m_1+m_b} \exp |\operatorname{Im} \lambda| \left( (b - \xi_n) \rho_n + \sum_{i=1}^n (\xi_i - \xi_{i-1}) \rho_{i-1} \right), & \deg b_1(\lambda) = \deg b_2(\lambda) \end{cases}$$

for  $\deg a_1(\lambda) = \deg a_2(\lambda)$ ;

$$|\Delta(\lambda)| \geq C_\delta \begin{cases} |\lambda|^{A+m+m_4} \exp |\operatorname{Im} \lambda| \left( (b - \xi_n) \rho_n + \sum_{i=1}^n (\xi_i - \xi_{i-1}) \rho_{i-1} \right), & \deg b_2(\lambda) > \deg b_1(\lambda) \\ |\lambda|^{A+m+m_3} \exp |\operatorname{Im} \lambda| \left( (b - \xi_n) \rho_n + \sum_{i=1}^n (\xi_i - \xi_{i-1}) \rho_{i-1} \right), & \deg b_1(\lambda) > \deg b_2(\lambda) \\ |\lambda|^{A+m+m_b} \exp |\operatorname{Im} \lambda| \left( (b - \xi_n) \rho_n + \sum_{i=1}^n (\xi_i - \xi_{i-1}) \rho_{i-1} \right), & \deg b_1(\lambda) = \deg b_2(\lambda) \end{cases}$$

in  $F_\delta \cap \tilde{F}_\delta$ , we can easily see that functions  $P_{ij}(x, \lambda)$  are bounded with respect to  $\lambda$ . As a result, these functions don't depend on  $\lambda$ .

Here, we denote  $\tilde{F}_\delta = \left\{ \lambda : |\lambda - \tilde{\lambda}_n| \geq \delta, n = 0, \pm 1, \pm 2, \dots \right\}$  where  $n$  is sufficiently small number,  $\tilde{\lambda}_n$  are eigenvalues of the problem  $\tilde{L}$ .

From (12),

$$\begin{aligned} P_{11}(x, \lambda) - 1 &= \frac{\tilde{\psi}_2(x, \lambda) (\varphi_1(x, \lambda) - \tilde{\varphi}_1(x, \lambda))}{\tilde{\Delta}(\lambda)} - \tilde{\varphi}_2(x, \lambda) \left( \frac{\psi_1(x, \lambda)}{\Delta(\lambda)} - \frac{\tilde{\psi}_1(x, \lambda)}{\tilde{\Delta}(\lambda)} \right) \\ P_{12}(x, \lambda) &= \frac{\psi_1(x, \lambda) (\tilde{\varphi}_1(x, \lambda) - \varphi_1(x, \lambda))}{\Delta(\lambda)} + \varphi_1(x, \lambda) \left( \frac{\psi_1(x, \lambda)}{\Delta(\lambda)} - \frac{\tilde{\psi}_1(x, \lambda)}{\tilde{\Delta}(\lambda)} \right) \\ P_{21}(x, \lambda) &= \varphi_2(x, \lambda) \left( \frac{\tilde{\psi}_2(x, \lambda)}{\tilde{\Delta}(\lambda)} - \frac{\psi_2(x, \lambda)}{\Delta(\lambda)} \right) + \psi_2(x, \lambda) \left( \frac{\varphi_2(x, \lambda) - \tilde{\varphi}_2(x, \lambda)}{\Delta(\lambda)} \right) \\ P_{22}(x, \lambda) - 1 &= \frac{\psi_2(x, \lambda) (\tilde{\varphi}_1(x, \lambda) - \varphi_1(x, \lambda))}{\Delta(\lambda)} + \varphi_2(x, \lambda) \left( \frac{\psi_1(x, \lambda)}{\Delta(\lambda)} - \frac{\tilde{\psi}_1(x, \lambda)}{\tilde{\Delta}(\lambda)} \right) \end{aligned}$$

It follows from the representations of solutions  $\varphi(x, \lambda)$  and  $\psi(x, \lambda)$ ,

$$\lim_{\lambda \rightarrow \infty} \frac{\tilde{\psi}_2(x, \lambda) (\varphi_1(x, \lambda) - \tilde{\varphi}_1(x, \lambda))}{\tilde{\Delta}(\lambda)} = 0 \text{ and}$$

$$\lim_{\lambda \rightarrow \infty} \tilde{\varphi}_2(x, \lambda) \left( \frac{\psi_1(x, \lambda)}{\Delta(\lambda)} - \frac{\tilde{\psi}_1(x, \lambda)}{\tilde{\Delta}(\lambda)} \right) = 0 \text{ for all } x \text{ in } \Lambda. \text{ Thus,}$$

$\lim_{\lambda \rightarrow \infty} (P_{11}(x, \lambda) - 1) = 0$  is valid uniformly with respect to  $x$ . So we have  $P_{11}(x, \lambda) \equiv 1$  and similarly  $P_{12}(x, \lambda) \equiv 0$ ,  $P_{21}(x, \lambda) \equiv 0$ ,  $P_{22}(x, \lambda) \equiv 1$ .

From (10), we obtain  $\varphi_1(x, \lambda) \equiv \tilde{\varphi}_1(x, \lambda)$ ,  $\Phi_1 \equiv \tilde{\Phi}_1$ ,  $\varphi_2(x, \lambda) \equiv \tilde{\varphi}_2(x, \lambda)$  and  $\Phi_2 \equiv \tilde{\Phi}_2$  for all  $x$  and  $\lambda$ . Moreover, from  $\Phi(x, \lambda) = \frac{\psi(x, \lambda)}{\Delta(\lambda)}$ , we get

$$\frac{\psi_2(x, \lambda)}{\psi_1(x, \lambda)} = \frac{\tilde{\psi}_2(x, \lambda)}{\tilde{\psi}_1(x, \lambda)}. \text{ Hence, } \Omega(x) = \tilde{\Omega}(x), \text{ i.e., } p(x) = \tilde{p}(x), r(x) = \tilde{r}(x) \text{ almost}$$

everywhere. On the other hand, since  $\begin{pmatrix} \varphi_{01}(a, \lambda) \\ \varphi_{02}(a, \lambda) \end{pmatrix} = \begin{pmatrix} a_2(\lambda) \\ a_1(\lambda) \end{pmatrix}$ ,  $\begin{pmatrix} \psi_{n1}(b, \lambda) \\ \psi_{n2}(b, \lambda) \end{pmatrix} = \begin{pmatrix} b_2(\lambda) \\ b_1(\lambda) \end{pmatrix}$ , we get easily that  $f_1(\lambda) = \tilde{f}_1(\lambda)$ ,  $f_2(\lambda) = \tilde{f}_2(\lambda)$ ,  $\varepsilon_i = \tilde{\varepsilon}_i$ ,  $\theta_i = \tilde{\theta}_i$ ,  $\gamma_i(\lambda) = \tilde{\gamma}_i(\lambda)$ ,  $i = \overline{1, n}$ . Therefore,  $L \equiv \tilde{L}$ .

**Theorem 2** If  $\lambda_n = \tilde{\lambda}_n$  and  $\mu_n = \tilde{\mu}_n$  for all  $n$ , then  $L \equiv \tilde{L}$ , i.e.,  $\Omega(x) = \tilde{\Omega}(x)$ , a.e.,  $f_1(\lambda) = \tilde{f}_1(\lambda)$ ,  $f_2(\lambda) = \tilde{f}_2(\lambda)$ ,  $\gamma_i(\lambda) = \tilde{\gamma}_i(\lambda)$ ,  $\varepsilon_i = \tilde{\varepsilon}_i$ ,  $\theta_i = \tilde{\theta}_i$ ,  $i = \overline{1, n}$ . Hence, the problem (1)-(5) is uniquely determined by spectral data  $\{\lambda_n, \mu_n\}$ .

**Proof** If  $\lambda_n = \tilde{\lambda}_n$  and  $\mu_n = \tilde{\mu}_n$  for all  $n$ , then  $M(\lambda) = \tilde{M}(\lambda)$ . Therefore, we get  $L = \tilde{L}$  by Theorem 1.

Let us consider the boundary-value problem  $L_1$  with the condition  $y_1(a) = 0$  instead of the condition (2) in  $L$ . Let  $\{\tau_n\}_{n \in \mathbb{Z}}$  be the eigenvalues of the problem  $L_1$ . It is clear that  $\tau_n$  are zeros of  $\Delta_1(\tau) := \psi_{n1}(a, \tau)$  which is characteristic function of  $L_1$ .

**Theorem 3** If  $\lambda_n = \tilde{\lambda}_n$  and  $\tau_n = \tilde{\tau}_n$  for all  $n$ , then  $L(\Omega, f_2(\lambda)) = L(\tilde{\Omega}, \tilde{f}_2(\lambda))$ .

Hence, the problem  $L$  is uniquely determined by the sequences  $\{\lambda_n\}$  and  $\{\tau_n\}$  except coefficients  $f_1(\lambda)$ .

**Proof** Since the characteristic functions  $\Delta(\lambda)$  and  $\Delta_1(\tau)$  are entire of order 1, functions  $\Delta(\lambda)$  and  $\Delta_1(\tau)$  are uniquely determined up to multiplicative constant with their zeros by Hadamard's factorization theorem [46]

$$\Delta(\lambda) = C \prod_{n=-\infty}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right),$$

$$\Delta_1(\tau) = C_1 \prod_{n=-\infty}^{\infty} \left(1 - \frac{\tau}{\tau_n}\right),$$

where  $C$  and  $C_1$  are constants depend on  $\{\lambda_n\}$  and  $\{\tau_n\}$ , respectively. When  $\lambda_n = \tilde{\lambda}_n$  and  $\tau_n = \tilde{\tau}_n$  for all  $n$ ,  $\Delta(\lambda) \equiv \tilde{\Delta}(\lambda)$  and  $\Delta_1(\tau) \equiv \tilde{\Delta}_1(\tau)$ . Hence,  $\psi_{n1}(a, \tau) = \tilde{\psi}_{n1}(a, \tau)$ . As a result, we get  $M(\lambda) = \tilde{M}(\lambda)$ . So, the proof is completed by Theorem 3.

## REFERENCES

- [1] Levitan B. M. and Sargsyan I. S., Sturm-Liouville and Dirac Operators [in Russian], Nauka (Moscow, 1988).
- [2] Berezanskii Yu. M., "Uniqueness theorem in the inverse spectral problem for the Schrödinger equation", Tr. Mosk. Mat. Obshch., 7, 3-51 (1958).
- [3] Gasymov M. G. and Dzhabiev T. T., Determination of a system of Dirac differential equations using two spectra, in Proceedings of School-Seminar on the Spectral Theory of Operators and Representations of Group Theory [in Russian] (Elm, Baku, 1975), pp. 46-71.
- [4] Marchenko V. A., Sturm-Liouville Operators and Their Applications [in Russian], Naukova Dumka, Kiev (1977).
- [5] Nizhnik L. P., Inverse Scattering Problems for Hyperbolic Equations [in Russian], Naukova Dumka, Kiev (1977).
- [6] Gasymov M. G., Inverse problem of the scattering theory for Dirac system of order  $2n$ , Tr. Mosk. Mat. Obshch., 19 (1968), 41-112; Birkhauser (Basel, 1997).



[7] Guseinov I. M., On the representation of Jost solutions of a system of Dirac differential equations with discontinuous coefficients, *Izv. Akad. Nauk Azerb. SSR*, 5 (1999), 41-45.

[8] Fulton C. T., Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions, *Proc. Roy. Soc. Edin.* 77A, pp. 293-308, 1977.

[9] Shkalikov A. A., Boundary Value Problems For Ordinary Differential Equations with a Parameter in Boundary Conditions *Trudy Sem. Imeny I. G. Petrovskogo*, 9, pp. 190-229, 1983.

[10] Yakubov S, Completeness of Root Functions of Regular Differential Operators, Logman, Scientific and Technical, New York, 1994.

[11] Kerimov NB, Memedov KhK, On a boundary value problem with a spectral parameter in the boundary conditions, *Sibir. Matem. Zhurnal.*, 40 (2), pp. 325-335, 1999 English translation: *Siberian Mathematical Journal*, 40 (2), pp. 281-290, 1999.

[12] Binding P. A., Browne P. J. and Watson B. A., Sturm-Liouville problems with boundary conditions rationally dependent on the eigenparameter II. *Journal of Computational and Applied Mathematics*, 148, pp. 147-169, 2002.

[13] Mukhtarov OSh, Kadakal M, Muhtarov FS. On discontinuous Sturm-Liouville problem with transmission conditions, *Journal of Mathematics of Kyoto University*, 444, pp. 779-798, 2004.

[14] Tunç E, Muhtarov OSh. Fundamental solution and eigenvalues of one boundary value problem with transmission conditions, *Applied Mathematics and Computation*, 157, pp. 347-355, 2004.

[15] Akdoğan Z, Demirci M, Mukhtarov OSh, Sturm-Liouville problems with eigendependent boundary and transmissions conditions, *Acta Mathematica Scientia*, 25B (4) pp. 731-740, 2005.

[16] Akdoğan Z, Demirci M, Mukhtarov OSh, Discontinuous Sturm-Liouville problem with eigenparameter-dependent boundary and transmission conditions, *Acta Applicandae Mathematicae*, 86 pp. 329-344, 2005.

[17] Fulton C. T., Singular eigenvalue problems with eigenvalue parameter contained in the boundary conditions, *Proc. R. Soc. Edinb. A.* 87, pp. 1-34, 1980.

[18] Amirov R. Kh., Ozkan A. S. and Keskin B., Inverse problems for impulsive Sturm-Liouville operator with spectral parameter linearly contained in boundary conditions, *Integral Transforms and Special Functions*, 20 (2009), 607-618.

[19] Guliyev N. J., Inverse eigenvalue problems for Sturm-Liouville equations with spectral parameter linearly contained in one of the boundary conditions, *Inverse Problems*, 21 (2005), 1315-1330.

[20] Mukhtarov O. Sh., Discontinuous boundary value problem with spectral parameter in boundary conditions, *Turkish J. Math.*, 18 (1994), 183-192.

[21] Russakovskii E. M., Operator treatment of boundary problems with spectral parameters entering via polynomials in the boundary conditions, *Funct. Anal. Appl.* 9, pp. 358-359, 1975.

[22] Binding P. A., Browne P. J. and Seddighi K., Sturm-Liouville problems with eigenparameter dependent boundary conditions, *Proc. Edinburgh Math. Soc.*, (2), 37 (1993), 57-72.

[23] Russakovskii E. M., Matrix boundary value problems with eigenvalue dependent boundary conditions, *Oper. Theory Adv. Appl.* (1995).

- [24] Mennicken R., Schmid H. and Shkalikov A. A., On the eigenvalue accumulation of Sturm-Liouville problems depending nonlinearly on the spectral parameter, *Math. Nachr.*, 189 (1998), 157-170.
- [25] Binding P. A., Browne P. J. and Watson B. A., Inverse spectral problems for Sturm-Liouville equations with eigenparameter dependent boundary conditions, *J. London Math. Soc.*, 62 (2000), 161-182.
- [26] Schmid H. and Tretter C., Singular Dirac systems and Sturm-Liouville problems nonlinear in the spectral parameter, *J. Differen. Equations* 181 (2) pp. 511-542, 2002.
- [27] Binding P. A., Browne P. J. and Watson B. A., Equivalence of inverse Sturm-Liouville problems with boundary conditions rationally dependent on the eigenparameter, *J. Math. Anal. Appl.*, 29 (2004), 246-261.
- [28] Hald O. H., Discontinuous inverse eigenvalue problems, *Comm. Pure Appl. Math.*, 37, 539-577 (1984).
- [29] Kobayashi, M., A uniqueness proof for discontinuous inverse Sturm-Liouville problems with symmetric potentials, *Inverse Problem*, 5, No. 5, 767-781 (1989).
- [30] Shepelsky D., The inverse problem of reconstruction of the medium's conductivity in a class of discontinuous and increasing functions, *Spectral Oper. Theory Rel. Topics: Adv. Sov. Math.*, 19, 209-232 (1994).
- [31] Amirov R. Kh., Güldü Y., Inverse Problems For Dirac Operator With Discontinuity Conditions Inside An Interval, *International Journal of Pure and Applied Mathematics*, 37, No. 2, 2007, 215-226.
- [32] Likov A. V. and Mikhailov Yu. A., *The Theory of Heat and Mass Transfer* Qosenergaizdat, 1963 (Russian).
- [33] Meschanov V.P. and Feldstein A. L., *Automatic Design of Directional Couplers*, Sviaz, Moscow, 1980.
- [34] Tikhonov A. N. and Samarskii A. A., *Equations of Mathematical Physics*, Pergamon, Oxford, 1990.
- [35] McLaughlin J. and Polyakov P., On the uniqueness of a spherical symmetric speed of sound from transmission eigenvalues, *J. Differ. Eqns* 107 (1994), pp. 351-382.
- [36] Voitovich N. N., Katsenelbaum B. Z. and Sivov A. N., *Generalized Method of Eigen-vibration in the Theory of Diffraction*, Nauka, Moskov, 1997 (in Russian).
- [37] Titeux I, Yakubov Ya., Completeness of root functions for thermal conduction in a strip with peicewise continuous coefficients, *Mathematical Models and Methods in Applied Sciences* 1997; 7 1035-1050.
- [38] Yurko V. A., Integral transforms connected with discontinuous boundary value problems, *Integral Transforms Spec. Funct.* 10 (2000), pp. 141-164.
- [39] Freiling G. and Yurko V. A., *Inverse Sturm-Liouville Problems and Their Applications*, Nova Science, New York, 2001.
- [40] Kadakal M., Mukhtarov O. Sh., Sturm-Liouville problems with discontinuities at two points, *Comput. Math. Appl.* 54 (2007) 1367-1379.
- [41] Yang Qiuxia and Wang Wanyi , Asymptotic behavior of a differential operator with discontinuities at two points, *Mathematical Methods in the Applied Sciences*, 34 pp. 373-383, 2011.
- [42] Shahriari Mohammad , Akbarfam Aliasghar Jodayree , Teschl Gerald , Uniqueness for inverse Sturm-Liouville problems with a finite number of transmission conditions, *J. Math. Anal. Appl.* 395 (2012) 19-29.

[43] Güldü Y., Inverse eigenvalue problems for a discontinuous Sturm-Liouville operator with two discontinuities, *Boundary Value Problems* 2013, 2013:209.

[44] Yang Chuan-Fu , Uniqueness theorems for differential pencils with eigenparameter boundary conditions and transmission conditions, *J. Differential Equations* 255 (2013) 2615–2635.

[45] Zhdanovich V. F., Formulae for the zeros of Dirichlet polynomials and quasipolynomials, *Dokl. Acad. Nauk SSSR* 135 (8) (1960), pp. 1046-1049.

[46] Titchmarsh E. C., *The Theory of Functions*, Oxford University Press, London (1939).

(Yalçın GÜLDÜ) , [MERVE ARSLANTAŞ] DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, CUMHURİYET UNIVERSITY, 58140 SIVAS, TURKEY

*E-mail address:* yguldu@gmail.com, merveguray0@gmail.com